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Rediscussion on Black Hole Angular Momentum

Jingyi Zhang

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Abstract An exact formula calculating a stationary axisymmetric black hole's angular momentum is given. As a check on the formula, the Kerr black hole's angular momentum is calculated. To obtain a correct result, we do the integration computating the angular momentum over both the event horizon's 2-dimensional section ∂B_+ and the inner horizon's 2-dimensional section ∂B_-, which is different from the definition of the original formula proposed by Bardeen and others. The angular momentums of two other categories of stationary axisymmetric non-Kerr black holes are calculated too. Because of the existing of the naked singularities, their angular momentums either go to infinity or can not be well-defined. These results indicate that the black hole thermodynamic first law can not be set up in these stationary axisymmetric non-Kerr black hole spacetimes. Without the Cosmic Censor Conjecture, the thermodynamic laws exclude these black holes.

Keywords Black hole · Angular momentum · Black hole thermodynamics · Cosmic Censor Conjecture

1 Introduction

From 1989 to 1993, Gutsunaev, Castejon-Amendo and Chamorro sequentially obtained a series of asymptotically flat, stationary axisymmetric non-Kerr black hole solutions [1–4] of the Einstein equations. The common character of these sorts of black hole solutions is that there are naked singularities which lie either on the stationary limit surface or on the event horizon surface. Thus, their stationary limit surfaces or event horizons are not topological sphere surfaces. By further study, one find that because of the existing of naked singularities, the dragging velocities of these black holes either vanish or become negative, but they still have angular momentum. Hence, what is the relationship between the dragging velocity and the angular momentum? Is there an angular momentum for a black hole whose dragging velocity vanishes? In this paper, we first derive an exact formula to calculate the angular

J. Zhang (✉)
Center for Astrophysics, Guangzhou University, 510006 Guangzhou, China
e-mail: physicz@tom.com

momentum. Then as a check on the formula, we calculate the angular momentums of the Kerr black hole and other two categories of stationary axisymmetric non-Kerr black holes. Finally, we have a discussion on our calculation results. Our derivation is partly based on [5, 6].

2 An Exact Angular Momentum Formula

As is well known, the general stationary axisymmetric black hole spacetime can be described by the line element

$$ds^2 = g_{00}dt^2 + g_{11}dx^2 + g_{22}dy^2 + g_{33}d\varphi^2 + 2g_{03}dtd\varphi. \tag{1}$$

There are a unique time translational Killing vector $(\frac{\partial}{\partial t})^a$ and a unique rotational Killing vector $(\frac{\partial}{\partial \varphi})^a$ whose orbits are closed curves with parameter length 2π. According to [7], the total angular momentum of a stationary axisymmetric black hole is defined as

$$J = -\frac{1}{8\pi}\int_{\partial B} K^{a;b} l_{[a} n_{b]} dA, \tag{2}$$

where $K^a = (\frac{\partial}{\partial \varphi})^a$. ∂B is a 2-dimensional section of the event horizon intersecting with a hypersurface S, which is spacelike, asymptotically flat and tangent to the rotation Killing vector K^a. l^a and n^b are two null tetrad vectors. Equation (2) requires that l^a is tangent to the generators of the horizon, and n^b orthogonal to ∂B. That is, on the surface ∂B we require

$$l^a \to \left(\frac{\partial}{\partial t}\right)^a + \Omega_H \left(\frac{\partial}{\partial \varphi}\right)^a, \tag{3}$$

$$n^a l_a = -1, \tag{4}$$

where Ω_H is the dragging angular velocity of the black hole and is the same at all points of the horizon. To obtain an exact formula calculating the angular momentum of a black hole, no loss of generalization, we set

$$l^a = \left(\frac{\partial}{\partial t}\right)^a + \alpha\left(\frac{\partial}{\partial x}\right)^a + \Omega\left(\frac{\partial}{\partial \varphi}\right)^a, \tag{5}$$

$$n^b = \beta\left(\frac{\partial}{\partial t}\right)^b - \frac{1}{2g_{11}\alpha}\left(\frac{\partial}{\partial x}\right)^b + \gamma\left(\frac{\partial}{\partial \varphi}\right)^b, \tag{6}$$

where $\Omega = -g_{03}/g_{33}$, α, β and γ are coefficients to be determined. From (3) we expect that when r approach to r_H, α vanishes and $1/2g_{11}\alpha$ is finite. The null tetrad vectors satisfy the pseudo-orthonormal relations

$$l^a l_a = 0, \qquad n^a n_a = 0, \qquad l^a n_a = -1, \tag{7}$$

and the metric condition

$$g^{ab} = -l^a n^b - n^a l^b + m^a \overline{m}^b + \overline{m}^a m^b. \tag{8}$$

From (7) and (8) and considering (3), we construct the null tetrad as follows:

$$l^a = \left(\frac{\partial}{\partial t}\right)^a + \sqrt{\frac{g_{03}^2 - g_{00}g_{33}}{g_{11}g_{33}}}\left(\frac{\partial}{\partial x}\right)^a - \frac{g_{03}}{g_{33}}\left(\frac{\partial}{\partial \varphi}\right)^a, \tag{9}$$

$$\begin{aligned} n^b = {} & \frac{g_{33}}{2(g_{03}^2 - g_{00}g_{33})}\left(\frac{\partial}{\partial t}\right)^b - \frac{1}{2g_{11}}\sqrt{\frac{g_{11}g_{33}}{g_{03}^2 - g_{00}g_{33}}}\left(\frac{\partial}{\partial x}\right)^b \\ & - \frac{g_{03}}{2(g_{03}^2 - g_{00}g_{33})}\left(\frac{\partial}{\partial \varphi}\right)^b, \end{aligned} \tag{10}$$

$$m^c = \sqrt{\frac{1}{2g_{22}}}\left(\frac{\partial}{\partial y}\right)^c + i\sqrt{\frac{1}{2g_{33}}}\left(\frac{\partial}{\partial \varphi}\right)^c. \tag{11}$$

Obviously, from (9), (10) and (11) we find that at the event horizon l^a satisfies condition (3), that is, α vanishes but $1/2g_{11}\alpha$ is finite.

With above null tetrad, we derive an exact formula calculating the total angular momentum. Because $K^{a;b}$ is an antisymmetric tensor, (2) can be written as

$$J = -\frac{1}{8\pi}\int_{\partial B} K^a_{;b} l_a n^b dA. \tag{12}$$

As a coordinate basis vector, K^a satisfies

$$K^a_{;b} = K^a_{,b} + \Gamma^a_{cb}K^c = \Gamma^a_{cb}K^c. \tag{13}$$

Therefore,

$$\begin{aligned} J &= -\frac{1}{8\pi}\int_{\partial B} \Gamma^a_{cb}K^c l_a n^b dA \\ &= -\frac{1}{8\pi}\int_{\partial B} \Gamma^a_{3b} g_{af} l^f n^b dA. \end{aligned} \tag{14}$$

Substituting (3) into (14) yields

$$J = J_s + J_u, \tag{15}$$

where

$$J_s := -\frac{1}{8\pi}\int_{\partial B} \Gamma^a_{3b} g_{a0} n^b dA, \tag{16}$$

$$J_u := -\frac{\Omega_H}{8\pi}\int_{\partial B} \Gamma^a_{3b} g_{a3} n^b dA. \tag{17}$$

Let us simplify (16) first. J_s can be written as

$$J_s = -\frac{1}{8\pi}\int_{\partial B} (\Gamma^0_{3b} g_{00} + \Gamma^3_{3b} g_{30}) n^b dA. \tag{18}$$

Since the spacetime that we discussed is stationary axisymmetric, namely, $g_{ab,0} = g_{ab,3} = 0$, we have

$$\Gamma^0_{30} = \Gamma^0_{33} = \Gamma^3_{30} = \Gamma^3_{33} = 0. \tag{19}$$

Substituting (6) into (18) and considering (19), we get

$$J_s = \frac{1}{8\pi}\int_{\partial B}(\Gamma^0_{31}g_{00} + \Gamma^3_{31}g_{30})\frac{1}{2g_{11}\alpha}dA. \tag{20}$$

By expressing Γ^0_{31} and Γ^3_{31} in the components of metric, we further simplify above expression of J_s as follows:

$$J_s = \frac{1}{32\pi}\int_{\partial B}\frac{1}{g_{11}\alpha}g_{03,1}dA. \tag{21}$$

Comparing (6) with (10) we get the expression of α. Namely,

$$\alpha = \sqrt{\frac{g_{03}^2 - g_{00}g_{33}}{g_{11}g_{33}}}. \tag{22}$$

Substituting (22) into (21), we finally obtain the exact formula of J_s. That is,

$$J_s = \frac{1}{32\pi}\int_{\partial B}\frac{g_{03,1}}{g_{11}}\sqrt{\frac{g_{11}g_{33}}{g_{03}^2 - g_{00}g_{33}}}dA. \tag{23}$$

Doing the similar derivations as above, we can also get

$$J_u = \frac{\Omega_H}{32\pi}\int_{\partial B}\frac{g_{33,1}}{g_{11}}\sqrt{\frac{g_{11}g_{33}}{g_{03}^2 - g_{00}g_{33}}}dA. \tag{24}$$

Finally, adding (23) and (24) yields the desired formula

$$J = \frac{1}{32\pi}\left(\int_{\partial B}\frac{g_{03,1}}{g_{11}}\sqrt{\frac{g_{11}g_{33}}{g_{03}^2 - g_{00}g_{33}}}dA + \Omega_H\int_{\partial B}\frac{g_{33,1}}{g_{11}}\sqrt{\frac{g_{11}g_{33}}{g_{03}^2 - g_{00}g_{33}}}dA\right). \tag{25}$$

Obviously, if $\Omega_H = 0$, we then have

$$J = J_s = \frac{1}{32\pi}\int_{\partial B}\frac{g_{03,1}}{g_{11}}\sqrt{\frac{g_{11}g_{33}}{g_{03}^2 - g_{00}g_{33}}}dA. \tag{26}$$

That is to say, maybe there are some black holes whose dragging angular velocities vanish, but they still have angular momentums.

3 Three Examples

3.1 Kerr Black Holes ($\Omega_H > 0$)

The Kerr black hole metric is

$$\begin{aligned} ds^2 = {} & -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left[(r^2 + a^2)\sin^2\theta + \frac{2Mra^2\sin^4\theta}{\rho^2}\right]d\varphi^2 \\ & - \frac{4Mra\sin^2\theta}{\rho^2}dtd\varphi, \end{aligned} \tag{27}$$

where

$$\rho \equiv r^2 + a^2 \cos^2\theta, \tag{28}$$

$$\Delta \equiv r^2 + a^2 - 2Mr. \tag{29}$$

It has two horizons defined by the hypersurfaces

$$r_\pm = M \pm \sqrt{M^2 - a^2}. \tag{30}$$

Using (25) and doing the integral over the event horizon's 2-dimensional section ∂B_+, we get

$$\begin{aligned} J_+ &= J_{s_+} + J_{u_+} \\ &= \left(\frac{M^4}{2a^2}\arctan\frac{a}{r_+} + \frac{M^2(a^2 - r_+^2)}{8ar_+}\right) \\ &\quad + \left(-\frac{M^4}{2a^2}\arctan\frac{a}{r_+} + \frac{M(9Mr_+^2 - Ma^2 - 4r_+^3)}{8ar_+}\right) \\ &= \frac{1}{2}Ma. \end{aligned} \tag{31}$$

It is half of the total angular momentum of the Kerr black hole. To obtain the correct result, let us do integral in (25) over inner horizon's 2-dimensional section ∂B_-. It gives

$$\begin{aligned} J_- &= J_{s_-} + J_{u_-} \\ &= \left(\frac{M^4}{2a^2}\arctan\frac{a}{r_-} + \frac{M^2(a^2 - r_-^2)}{8ar_-}\right) \\ &\quad + \left(-\frac{M^4}{2a^2}\arctan\frac{a}{r_-} + \frac{M(9Mr_-^2 - Ma^2 - 4r_-^3)}{8ar_-}\right) \\ &= \frac{1}{2}Ma. \end{aligned} \tag{32}$$

Obviously, if we consider that the 2-dimensional boundary of the black hole consists of the event horizon's 2-dimensional section ∂B_+ and the inner horizon's 2-dimensional section ∂B_-, that is $\partial B = \partial B_+ + \partial B_-$, we can get the correct total angular momentum of the hole by doing integral in (25) over ∂B_+ and ∂B_-.

3.2 Black Holes with $\Omega_H = 0$

Reference [2] gives us a type of asymptotically flat, stationary axisymmetric black hole solution. The dragging angular velocity of the hole vanishes at the event horizon, which is different from the Kerr black hole. The line element can be written in prolate spheroidal coordinates as follows:

$$ds^2 = K^2 f^{-1}\left[e^{2\gamma}(x^2 - y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\varphi^2\right] - f(dt - \omega d\varphi)^2, \tag{33}$$

where

$$f = 2p(x^2-1)C/D, \qquad e^{2\gamma} = \frac{x^2-1}{x^2-y^2}\left[1 - \frac{q^2(x^2-1)^3(1-y^2)}{(x-y)^8}\right],$$
$$\omega = -2Kq(1-y^2)E/pC,$$
$$C := (x-y)^8 - q^2(x^2-1)^3(1-y^2),$$
$$D := (p+1)(x+1)^2[(x-y)^4 + (p-1)(x-1)^3(1+y)]^2 + (p-1)(x-1)^2[(x-y)^4 + (p+1)(x+1)^3(1-y)]^2,$$
$$E := (x-y)^5(3x^2 - 3xy + 3px - py + y^2 + 1) + q^2(x^2-1)^3(x - 2y + p), \tag{34}$$

p, q and K are three real parameters satisfying $p^2 - q^2 = 1$. If $q = 0$, (33) goes over to the Schwarzschild metric. The hypersurfaces $x = 1$ and $x = -1$ define locally the event horizon and inner horizon respectively. There are singularities on the stationary limit surface.

Since

$$\lim_{x\to 1} f = 0, \tag{35}$$

but

$$\lim_{x\to 1} K^2 f^{-1}(x^2-1)(1-y^2) \neq 0, \tag{36}$$

so

$$\Omega_H = \lim_{x\to 1} \Omega = 0. \tag{37}$$

Doing the integral in (26) over the event horizon's 2-dimensional section ∂B_+ and the inner horizon's 2-dimensional section ∂B_-, we get

$$J_{s_+} = -\frac{K^2 q}{4p}\int_{-1}^{1} \frac{(1+y)[y^2 - (3+p)y + 4 + 3p]}{(1-y)^2} dy, \tag{38}$$

$$J_{s_-} = -\frac{K^2 q}{4p}\int_{-1}^{1} \frac{(1-y)[y^2 + (3-p)y + 4 - 3p]}{(1+y)^2} dy. \tag{39}$$

Let $u = 1 + y$. Then we have

$$J = J_{s_+} + J_{s_-} = -\frac{2K^2 q}{p}\int_0^2 \frac{1}{u^2} du \to -\infty. \tag{40}$$

Similar calculations yield that the total angular momentum of the black holes given by [1, 3] are all divergent.

3.3 Black Holes with $\Omega_H < 0$

In [4], another type of asymptotically flat, stationary axisymmetric non-Kerr black hole solution is given. The difference with above type is that this type of black hole has an reverse dragging angular velocity ($\Omega_H < 0$), and the singularities lie on the event horizon. If we write the line element as (33) and replace K with M, the expressions of f, ω and γ can be written as follows:

$$
\begin{aligned}
& f = B/C, \qquad e^{2\gamma} = B[16M^4(x-y)(x+y)^9]^{-1}, \qquad \omega = -4M^2 a(x+1)(1-y^2)D/B, \\
& B := (x^2-1)[4M^2(x+y)^4 + a^2(1-y^2)^2]^2 - 4M^2a^2(1-y^2)[(x+y)^4 - (x^2-1)^2]^2, \\
& C := (x^2+1)[4M^2(x+y)^4 - a^2(1-y)(1+y)^3]^2 \\
& \quad + 4M^2a^2(1+y^2)[(x+y)^4 - (x-1)(x+1)^3]^2, \\
& D := 4M^2(x+y)^5[2x(x^2-1) - (1+y)(2xy+y^2+1)] \\
& \quad - a^2(1+y)^3[(x+y)^5 - (x-1)(x+1)^3(x-y)],
\end{aligned} \tag{41}
$$

where M is the total mass of the black holes, a represents the angular momentum per unit mass. The solution reduces to the Schwarzschild solution in the absence of rotation ($a = 0$). The event horizon locates at $x = 1$. There is no inner horizon in the spacetime. The dragging angular velocity of this black hole is negative, namely,

$$
\Omega_H = \lim_{x\to 1} \Omega = -\frac{a}{2(4M^2+a^2)}. \tag{42}
$$

That is, this type of hole has a reverse dragging velocity.

Using the formula (25) we get

$$
J_s = -\frac{(4M^2+a^2)^2}{4Ma^3} J_1 + \frac{(4M^2+a^2)^3}{2M^3a^5} J_2, \tag{43}
$$

$$
J_u = -\frac{M(4M^2+a^2)}{2a} + \frac{(4M^2+a^2)^2}{2Ma^3} J_1 - \frac{(4M^2+a^2)^3}{2M^3a^5} J_2 + \frac{(4M^2+a^2)^3}{4M^3a^5} J_3, \tag{44}
$$

where

$$
J_1 = \int_0^2 \frac{-(4M^2+a^2)u^6 - 8(6M^2+a^2)u^5 + 4(32M^2+5a^2)u^4 - 16(4M^2+a^2)u^2 + 64a^2u - 64a^2}{[u^4 + \frac{(4M^2+a^2)^2}{M^2a^2}u^2 - \frac{4(4M^2+a^2)}{M^2}u + \frac{4a^2}{M^2}]^2} du,
$$

$$
J_2 = \int_0^2 \frac{-8M^2a^2u^9 - (4M^2-a^2)^2u^8 - 6(16M^4-a^4)u^7 + 4(64M^4+28M^2a^2-3a^4)u^6 + (8a^4-160M^2a^2)u^5}{[u^4 + \frac{(4M^2+a^2)^2}{M^2a^2}u^2 - \frac{4(4M^2+a^2)}{M^2}u + \frac{4a^2}{M^2}]^3} du,
$$

$$
J_3 = 8a^2 \int_0^2 \frac{2M^2u^5 + (2M^4/a^2 + a^2/8 - 3M^2)u^4 - (4M^2+a^2)u^3 + (4M^2+3a^2)u^2 - 4a^2u + 2a^2}{[u^4 + \frac{(4M^2+a^2)^2}{M^2a^2}u^2 - \frac{4(4M^2+a^2)}{M^2}u + \frac{4a^2}{M^2}]^2} du.
$$

The total angular momentum is

$$
\begin{aligned}
J = J_s + J_u &= -\frac{M(4M^2+a^2)}{2a} + \frac{(4M^2+a^2)^2}{4Ma^3}\left(J_1 + \frac{4M^2+a^2}{M^2a^2} J_3\right) \\
&= -\frac{M(4M^2+a^2)}{2a}.
\end{aligned} \tag{45}
$$

It doesn't equal Ma. Moreover, if $a = 0$, $J \to -\infty$. It can not go over to the Schwarzschild case.

4 Conclusion and Discussion

We have calculated the total angular momentums of the Kerr black hole and other two types of stationary axisymmetric black holes. To obtain the correct result we should do the calculations over the event horizon's 2-dimensional section ∂B_+ and the inner horizon's 2-dimensional section ∂B_-. It means that when we investigate the thermal properties of a black hole, we should consider the contribution of the inner horizon as well as the event horizon. Moreover, our calculations indicate that because of the existing of the naked singularities, the total angular momentums of the non-Kerr black holes either go to infinity or can not be well-defined. That is, we can not define the total angular momentum of these types of black holes, thus we can not set up the black hole thermodynamic first law. In fact, the temperatures in two poles of these black hole event horizons either vanish or go to infinity [8]. It is the thermodynamic laws that exclude these stationary axisymmetric non-Kerr black holes.

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